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AN INFINITE FAMILY IN ${}_2\pi_*^S$ AT ADAMS FILTRATION SEVEN

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ABSTRACT. We prove the family $\{h_i^2 h_3 d_1\}$ in $\mathrm{Ext}_A^{7,*}(\mathbb{Z}_2, \mathbb{Z}_2)$ detects homotopy elements in the 2-adic stable homotopy of spheres ${}_2\pi_*^S$ where A is the mod 2 Steenrod algebra.

1. INTRODUCTION

In this paper we construct an infinite family in the 2-adic homotopy groups of spheres ${}_2\pi_*^S$ at Adams filtration 7. This family $\{h_i^2 h_3 d_1\}$ is to be described later. The construction is based on Mahowald's method [20] which makes use of some nice properties of the Brown-Gitler spectra [6]. By this method some infinite families at low Adams filtrations in ${}_2\pi_*^S$ and also in ${}_p\pi_*^S$ for odd primes p have been constructed [8, 10, 11, 15, 16, 20]. All of these families are of Adams filtration at most 4. The family $\{h_i^2 h_3 d_1\}$ has Adams filtration 7 which is a little bit higher. We need to make some calculations on the cohomology $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_2, \mathbb{Z}_2)$ of the mod 2 Steenrod algebra A in relevant dimensions in order to show the nontriviality of the family $\{h_i^2 h_3 d_1\}$. These Ext groups will be calculated by May spectral sequence [21].

We begin by recalling [1] that the mod 2 Adams spectral sequence for ${}_2\pi_*^S$ has $E_2^{s,t} \cong \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$ where A is the mod 2 Steenrod algebra. Let $h_i \in \mathrm{Ext}_A^{1,2^i}(\mathbb{Z}_2, \mathbb{Z}_2)$ be the class corresponding to the generator $\mathrm{Sq}^{2^i} \in A$ $(i \geq 0)$. It is known [2] that $h_i^2 \neq 0$ in $\mathrm{Ext}_A^{2,2^{i+1}}(\mathbb{Z}_2, \mathbb{Z}_2)$ for all $i \geq 0$. J. P. May [21] and M. C. Tangora [22] have shown that $\mathrm{Ext}_A^{4,36}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$ is generated by an element called d_1 and that there is a nonzero element $e_1 \in \mathrm{Ext}_A^{4,42}(\mathbb{Z}_2, \mathbb{Z}_2)$ such that $h_3 d_1 = h_1 e_1 \neq 0$ in $\mathrm{Ext}_A^{5,44}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$. It is proved in [14] that $h_i^2 h_3 d_1 = h_i^2 h_1 e_1 \neq 0$ in $\mathrm{Ext}_A^{7,2^{i+1}+44}(\mathbb{Z}_2, \mathbb{Z}_2)$ for $i > 12$. (The result might still be true for smaller values of i.)

Theorem 1.1. *For each $i > 12$, the class $h_i^2 h_3 d_1 = h_i^2 h_1 e_1$ in $\mathrm{Ext}_A^{7,2^{i+1}+44}(\mathbb{Z}_2, \mathbb{Z}_2)$ detects homotopy elements in $\pi_{2^{i+1}+37}^S$.*

We need to show the following.

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Proposition 1.2. *For each* $i > 12$, $h_i^2 h_3 d_1$ *is an infinite cycle in the* mod 2 *Adams spectral sequence for* ${}_2\pi_*^S$.

Proposition 1.3. *For each* $i > 12$, $h_i^2 h_3 d_1$ *is not a boundary in the* mod 2 *Adams spectral sequence for* ${}_2\pi_*^S$.

The proof of Proposition 1.2 is based upon Mahowald's method mentioned above, and also upon the following result.

Proposition 1.4. $\mathrm{Ext}_A^{s, 2^{i+1}+3+s}(\mathbb{Z}_2, \mathbb{Z}_2) = 0$ *for* $i > 12$ *and* $s \leq 5$.

Remark. Mahowald and Tangora [19] have shown that d_1 and hence $h_3 d_1 = h_1 e_1$ detect homotopy elements in ${}_2\pi_*^S$. So (1.2) is immediate if h_i^2 survives the Adams spectral sequence, but this is known true only for $i \leq 5$ so far [5, 19]. Note that e_1 does not detect homotopy elements, as erroneously claimed in [19], and corrected by R. Bruner in [9].

Proposition 1.3 follows from the following proposition. We recall [21, 22] that $\mathrm{Ext}_A^{3,11}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ is generated by an element called c_0 and that $\mathrm{Ext}_A^{3,42}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ and $\mathrm{Ext}_A^{4,43}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ are generated respectively by $h_5 h_3 h_1$ and $h_5 c_0$.

Proposition 1.5. *For* $i > 12$,

$$\mathrm{Ext}_A^{s, 2^{i+1}+38+s}(\mathbb{Z}_2, \mathbb{Z}_2) = \begin{cases} 0 & \textit{for } s \leq 3, \\ \mathbb{Z}_2 & \textit{generated by } h_{i+1} h_5 h_3 h_1 \textit{ for } s = 4, \\ \mathbb{Z}_2 & \textit{generated by } h_{i+1} h_5 c_0 \textit{ for } s = 5. \end{cases}$$

It is known that $d_3(h_5 h_3) = 0$ and $d_2(h_{i+1}) = h_i^2 h_0 \neq 0$ $(i \geq 4)$ in the Adams spectral sequence for ${}_2\pi_*^S$ [2, 19], that $h_{i+1} h_1$ $(i \geq 2)$ and $h_5 c_0$ detect homotopy elements [19, 20] and that $h_0 c_0 = 0$ in $\mathrm{Ext}_A^{4,12}(\mathbb{Z}_2, \mathbb{Z}_2)$ [22]. Then $d_3(h_{i+1} h_5 h_3 h_1) = 0$ and $d_2(h_{i+1} h_5 c_0) = 0$ in the Adams spectral sequence. This proves Proposition 1.3.

Proof of Proposition 1.2 is given in §2. Proofs of Propositions 1.4 and 1.5 are given in §3.

2. Proof of Proposition 1.2

All cohomology groups of a space or a spectrum to be considered have $\mathbb{Z}_2$ coefficients.

Let $\mathbb{R}P^n$ be the n-dimensional real projective space and let P^n denote its suspension spectrum. For $m \in \mathbb{Z}$, S^m will denote the sphere spectrum in stable dimension m.

For any $a, b \in \mathbb{Z}$ with $a < b$ there is a spectrum P_a^b which, when $a > 0$, is the suspension spectrum of the stunted projective space $\mathbb{R}P^b / \mathbb{R}P^{a-1}$. These can be defined as Thom spectra or by Jame's periodicity as in [3]. They have the following properties:

(1) P_a^b has a stable cell in each dimension from a to b.

(2) $H^*(P_a^b)$ has $\{x^a, x^{a+1}, \dots, x^b\}$ as a $\mathbb{Z}_2$-base with the Steenrod operations

$$\mathrm{Sq}^k x^i = \binom{2^m+i}{k} x^{i+k}$$

where m is any positive integer such that $2^m + i > k$.

(3) $P_0^b = S^0 \vee P^b$ for $b > 0$.

(4) $P_a^{-1} = P_a^{-2} \vee S^{-1}$ for $a < -2$.

(5) There are cofibration sequences

$$P_a^b \to P_a^c \to P_{b+1}^c \xrightarrow{g} \sum P_a^b \to \sum P_a^c \quad \text{for } a < b < c.$$

We will be interested in, for $i \geq 1$, the composite

$$\overline{g}: P^5 = P_1^5 \to P_1^5 \vee S^0 = P_0^5 \xrightarrow{g} \sum P_{-2^i-1}^{-1} = \sum P_{-2^i-1}^{-2} \vee S^0 \to \sum P_{-2^i-1}^{-2}$$

where the first map is the inclusion and the third map is the projection. The map we want to construct for Theorem 1.1 for each $i > 12$ is a composite of the form

$$S^{37} \xrightarrow{\{\overline{d_1}\}} S^3 \cup_\eta e^5 \xrightarrow{j} P^5 \xrightarrow{\overline{g}} \sum P_{-2^i-1}^{-2} \xrightarrow{f} S^{-2^{i+1}}$$

where η is the generator of $\pi_1^S = \mathbb{Z}_2$. These maps, except $\overline{g}$, are described as follows.

It is easy to see that there is a map $j: S^3 \cup_\eta e^5 \to P^5$ which is nonzero in cohomology in dimensions 3 and 5 (for reference see [18]). From (2) and (5) we have the following.

(6) In the mapping cone $X = \sum P_{-2^i-1}^{-2} \cup_{\overline{g}\circ j} e^4 \cup_\eta e^6$, the Steenrod operation

$$\mathrm{Sq}^8: H^{-2}(X) = \mathbb{Z}_2 \to H^6(X) = \mathbb{Z}_2$$

is nonzero.

Let $\{d_1\}$ denote any homotopy class in ${}_2\pi_{32}^S$ detected by $d_1 \in \mathrm{Ext}_A^{4,36}(\mathbb{Z}_2, \mathbb{Z}_2)$ in the Adams spectral sequence. The existence of such classes is proved in [19], as is the fact that any $\{d_1\}$ has order 2.

Lemma 2.1. *There is a* $\{d_1\}$ *such that* $\eta\{d_1\} = 0$.

We note [4] that $h_1 d_1 \neq 0$ in $\mathrm{Ext}_A^{5,38}(\mathbb{Z}_2, \mathbb{Z}_2)$ and that $d_3(h_2 h_5) = h_1 d_1$ in the Adams spectral sequence for ${}_2\pi_*^S$. The differential $d_3(h_2 h_5) = h_1 d_1$ suggests the result (2.1), but the latter cannot be simply concluded from the former alone. Some more work, though not difficult, is needed for the proof of (2.1), and this is done at the end of this section.

Let $\Phi_{2,5}$ be the secondary cohomology operation dual to $h_2 h_5$ as described Adams in [2]. From Lemma 2.1 and the fact that $d_3(h_2 h_5) = h_1 d_1$ we have the following.

(7) (i) There is a map $S^{37} \xrightarrow{\{\overline{d_1}\}} S^3 \cup_\eta e^5$ such that the diagram

$$\begin{array}{ccc} S^{37} & \xrightarrow{\{\overline{d_1}\}} & S^3 U_\eta e^5 \\ & {\scriptstyle \{d_1\}} \searrow \quad \swarrow {\scriptstyle p} & \\ & S^5 & \end{array}$$

is commutative where $\{d_1\}$ is as in (2.1) and p is the collapsing map.

(ii) In the mapping cone $Y = S^3 \cup_\eta e^5 \cup_{\{\overline{d_1}\}} e^{38}$, the secondary cohomology operation

$$\Phi_{2,5}: H^3(Y) = \mathbb{Z}_2 \to H^{38}(Y) = \mathbb{Z}_2$$

is nonzero with zero indeterminancy.

Proposition 2.2. *For each $i \geq 1$ there is a map $f: \sum P^{-2}_{-2^i-1} \to S^{-2^{i+1}}$ such that, in the mapping cone $Z = S^{-2^{i+1}} \cup_f C \sum P^{-2}_{-2^i-1}$, the Steenrod operation*

$$\mathrm{Sq}^{2^{i+1}}: H^{-2^{i+1}}(Z) = \mathbb{Z}_2 \to H^0(Z) = \mathbb{Z}_2$$

is nonzero.

The construction of such a map f is obtained via some nice properties of the Brown-Gitler spectra [6, 7]; it is primarily due to Mahowald [20]. We refer to [12] for a detailed proof of the result. A sketchy proof is given in [15].

Let $\Phi_{i,i}$ be the secondary cohomology operation dual to h_i^2 as described by Adams in [2]. Then from (2.2) and Adams' Hopf invariant one theorem we deduce the following.

(8) For $i \geq 3$, the secondary cohomology operation

$$\Phi_{i,i}: H^{-2^{i+1}}(Z) = \mathbb{Z}_2 \to H^{-1}(Z) = \mathbb{Z}_2$$

is nonzero with zero indeterminancy.

Now assume $i > 12$ and consider the composite

$$S^{37} \xrightarrow{\{\overline{d_1}\}} S^3 U_\eta e^5 \xrightarrow{j} P^5 \xrightarrow{\overline{g}} \sum P^{-2}_{-2^i-1} \xrightarrow{f} S^{-2^{i+1}}.$$

Since Sq^8 is dual to h_3 and $\Phi_{i,i}$ is dual to h_i^2 it follows from (6) and (8) that the composite

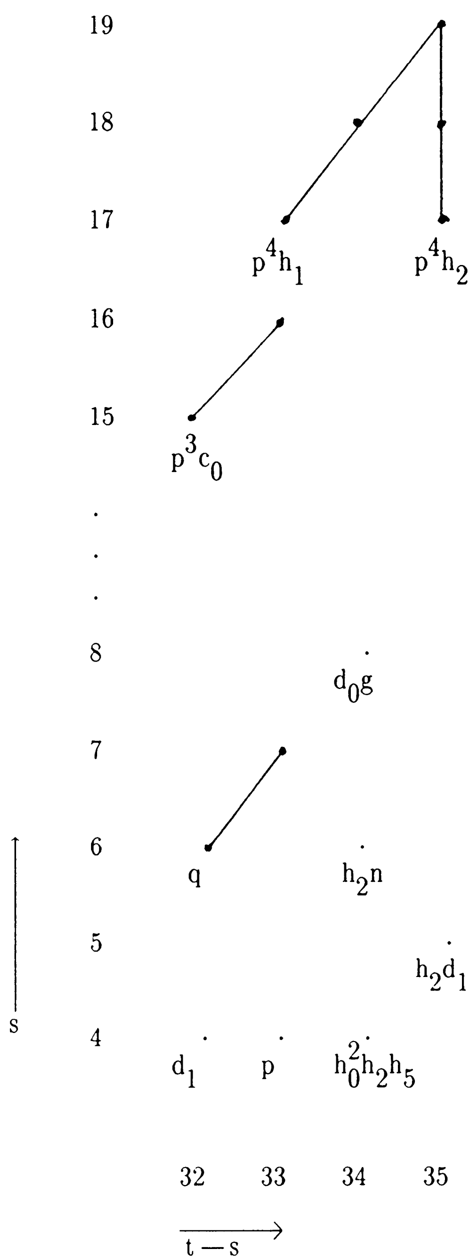
$$\varphi: S^3 \cup_\eta e^5 \xrightarrow{j} P^5 \xrightarrow{\overline{g}} \sum P^{-2}_{-2^i-1} \xrightarrow{f} S^{-2^{i+1}}$$

is detected by $h_i^2 h_3$ on the top cell e^5. By Proposition 1.4 $\varphi|S^3: S^3 \to S^{-2^{i+1}}$ has Adams filtration at least 6. Since $h_i^2 h_3 d_1$ is at Adams filtration 7, $h_2 h_5$ is at filtration 2 and $\Phi_{2,5}$ is dual to $h_2 h_5$, it follows from (7) that the composite

$$S^{37} \xrightarrow{\{\overline{d_1}\}} S^3 \cup_\eta e^5 \xrightarrow{\varphi} S^{-2^{i+1}}$$

is detected by $h_i^2 h_3 d_1$ provided it is not a boundary in the Adams spectral sequence. This proves Proposition 1.2.

It remains to prove Lemma 2.1. The following portion of the E_∞-term of the mod 2 Adams spectral sequence for ${}_2\pi_*^S$ is depicted from [19]. Here vertical (resp. diagonal) lines denote multiplication by h_0 (resp. h_1), corresponding to multiplication by 2 (resp. η) in homotopy, up to elements of higher filtration.



E_∞-chart

It is proved in [19] that ${}_2\pi^S_{32}$ is a vector space over $\mathbb{Z}_2$. In particular, any $\{d_1\}$ has order 2. Suppose $\eta\{d_1\} \neq 0$ for some $\{d_1\}$. From the E_∞-chart above we see either $\eta\{d_1\} = \eta\{q\}$ or $\eta\{d_1\} = \eta\{p^3c_0\}$ for some $\{q\}$ and $\{p^3c_0\}$, or $\eta\{d_1\}$ is detected by p^4h_1. Since $\{d_1\} + \{q\}$ or $\{d_1\} + \{p^3c_0\}$ is also a $\{d_1\}$, we need only show that the third case is impossible. Suppose $\eta\{d_1\}$ is detected by p^4h_1. Then $\eta^3\{d_1\}$ is detected by $h_1^2p^4h_1 \neq 0$ and therefore is nonzero. On the other hand, $\eta^3\{d_1\} = 4\nu\{d_1\} = 0$. This contradiction proves Lemma 2.1.

3. Proofs of Propositions 1.4 and 1.5

To begin with, we recall the following facts from [2 and 23].

Theorem 3.1 [2].

(i) $\mathrm{Ext}_A^{1,*}(\mathbb{Z}_2, \mathbb{Z}_2)$ *has* $\{h_k | k \geq 0\}$ *as a* $\mathbb{Z}_2$*-base where* $h_k \in \mathrm{Ext}_A^{1,2^k}(\mathbb{Z}_2, \mathbb{Z}_2)$.
(ii) $\mathrm{Ext}_A^{2,*}(\mathbb{Z}_2, \mathbb{Z}_2)$ *has* $\{h_kh_l | 0 \leq k \leq l, k \neq l-1\}$ *as a* $\mathbb{Z}_2$*-base.*
(iii) *If we take the products* $h_kh_lh_m$ *for* $0 \leq k \leq l \leq m$ *and remove the products*

$$h_kh_lh_{l+1}, \; h_kh_{k+1}h_m, \; h_kh_{k+2}^2, \; h_k^3 \qquad (k \geq 1),$$

then the remaining products are linearly independent in $\mathrm{Ext}_A^{3,*}(\mathbb{Z}_2, \mathbb{Z}_2)$.

Theorem 3.2 [23]. *For each* $k \geq 0$ *there is a nonzero class*

$$c_k \in \mathrm{Ext}_A^{3, 2^{k+3}+3\cdot 2^k}(\mathbb{Z}_2, \mathbb{Z}_2)$$

such that $\{c_k | k \geq 0\}$ *plus the linearly independent set in* (3.1)(iii) *is a* $\mathbb{Z}_2$*-base for* $\mathrm{Ext}_A^{3,*}(\mathbb{Z}_2, \mathbb{Z}_2)$.

Propositions 1.4 and 1.5 for $s \leq 3$ follow from these results. The remainder of this section is devoted to proving the propositions for $s = 4$ and $s = 5$.

Consider the polynomial algebra $X = \mathbb{Z}_2[R^i_j]$ over $\mathbb{Z}_2$ on the symbols R^i_j where $i, j \in \mathbb{Z}$ with $i \geq 0$, $j \geq 1$. We trigrade the algebra X by assigning to each generator R^i_j the tridegree $(1-j, j, 2^i(2^j-1))$. X is a differential algebra over $\mathbb{Z}_2$ with differential δ given on the generators by

$$\delta(R^i_j) = \sum_{k=1}^{j-1} R^{i+k}_{j-k} R^i_k. \tag{9}$$

The May spectral sequence [21] for computing $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_2, \mathbb{Z}_2)$ is a trigraded multiplicative spectral sequence with $E_2^{p,q,t} = H^{p,q,t}(X)$ and has

$$\bigoplus_{p+q=s} E_\infty^{p,q,t} \text{ isomorphic to } \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2) \text{ as a } \mathbb{Z}_2\text{-module.} \tag{10}$$

The degree s in $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$ is usually called the homological degree. We call t the internal degree; it is associated with the degree in the Steenrod algebra

A. Under the degree correspondence in (10) we see the length of the monomial $(R_{j_1}^{i_1})^{k_1}\cdots(R_{j_n}^{i_n})^{k_n}$, which is $k_1+\cdots+k_n$, corresponds to the homological degree and $\sum_{l=1}^{n} k_l 2^{i_l}(2^{j_l}-1)$ is the internal degree. For this reason we bigrade X by assigning R_j^i the bidegree $(1, 2^i(2^j-1))$ and still call s (resp. t) in $X^{s,t}$ the homological degree (resp. internal degree). Occasionally we will also call s the homological dimension of x if $x \in X^{s,*}$ or $x \in H^{s,*}(X)$.

It is easy to see that $h_i = \{R_1^i\}$ and $b_j^i = \{(R_j^i)^2\}$ for $j \geq 2$ are indecomposable elements in the cohomology algebra $H^{*,*}(X)$. May constructs in [21] some other indecomposables $h_i(S)$ for certain sequences S of positive integers, and these we describe as follows.

We will consider monomials $R_{j_1}^{i_1}\cdots R_{j_n}^{i_n}$ for $n \geq 2$ such that the set $\{i_1, i_2, \ldots, i_n, i_1+j_1, i_2+j_2, \ldots, i_n+j_n\}$ is equal to the set $\{i, i+1, \ldots, i+2n-1\}$ for some $i \geq 0$. Note that $i = i_k$ for a unique k. Let $\{i_{m_1}, \ldots, i_{m_{n-1}}\}$ be the set of the remaining i_l such that $i_{m_1} < i_{m_2} < \cdots < i_{m_{n-1}}$ and let S be the sequence of positive integers $(i_{m_1}-i_k, i_{m_2}-i_k, \ldots, i_{m_{n-1}}-i_k)$. Then we write $R_{j_1}^{i_1}\cdots R_{j_n}^{i_n} \in \overline{S}$. For $i \geq 0$ and any sequence S of positive integers of length $n-1$ which arises this way let $H_i(S)$ be the sum of all $R_{j_1}^{i_1}\cdots R_{j_n}^{i_n}$ such that $i = \min\{i_1, \ldots, i_n\}$ and $R_{j_1}^{i_1}\cdots R_{j_n}^{i_n} \in \overline{S}$. It is proved in [21] that each $H_i(S)$ is a cycle. Let $h_i(S) = \{H_i(S)\} \in H^{*,*}(X)$. Then May conjectures in [21] that $\{h_i(S) | S \text{ primitive}\} \cup \{h_i\} \cup \{b_j^i\}$ is a $\mathbb{Z}_2$-base for the indecomposable elements of $H^{*,*}(X)$. We refer to [17] for the notion of a "primitive" sequence S.

In [17] we have proved this conjecture up to homological dimension 4, and also in dimension 5 for some particular internal degrees. The proofs of Propositions 1.4 and 1.5 for $s = 4$ and $s = 5$ will be based on this result. To describe the result we note that the primitive sequences S for which $h_i(S)$ have homological dimensions ≤ 4 are (1), (1, 2), (1, 3), (1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 3, 4) and (1, 3, 5). These $h_i(S)$ are described as follows.

(11) $h_i(1) \in H^{2,*}(X)$, represented by

$$H_i(1) = R_1^{i+1}R_3^i + R_2^{i+1}R_2^i \qquad (i \geq 0).$$

$h_i(1, 2) \in H^{3,*}(X)$, represented by

$$H_i(1, 2) = R_1^{i+2}R_3^{i+1}R_5^i + R_2^{i+1}R_2^{i+2}R_5^i + R_1^{i+2}R_4^iR_4^{i+1} + R_3^iR_2^{i+2}R_4^{i+1} + R_2^{i+1}R_4^iR_3^{i+2} + R_3^iR_3^{i+1}R_3^{i+2} \qquad (i \geq 0).$$

$h_i(1, 3) \in H^{3,*}(X)$, represented by

$$H_i(1, 3) = R_1^{i+3}R_1^{i+1}R_5^i + R_1^{i+1}R_4^iR_2^{i+2} + R_2^iR_1^{i+3}R_4^{i+1} + R_2^iR_3^{i+1}R_2^{i+3} \qquad (i \geq 0).$$

$h_i(1, 2, 3) \in H^{4,*}(X)$, represented by

$$H_i(1, 2, 3) = R_1^{i+3}R_3^{i+2}R_5^{i+1}R_7^i + \overline{H}_i(1, 2, 3) \qquad (i \geq 0).$$

$h_i(1, 2, 4) \in H^{4,*}(X)$, represented by
$$H_i(1, 2, 4) = R_1^{i+4}R_1^{i+2}R_5^{i+1}R_7^i + \overline{H}_i(1, 2, 4) \qquad (i \geq 0).$$
$h_i(1, 2, 5) \in H^{4,*}(X)$, represented by
$$H_i(1, 2, 5) = R_1^{i+5}R_1^{i+2}R_3^{i+1}R_7^i + \overline{H}_i(1, 2, 5) \qquad (i \geq 0).$$
$h_i(1, 3, 4) \in H^{4,*}(X)$, represented by
$$H_i(1, 3, 4) = R_1^{i+4}R_3^{i+3}R_1^{i+1}R_7^i + \overline{H}_i(1, 3, 4) \qquad (i \geq 0).$$
$h_i(1, 3, 5) \in H^{4,*}(X)$, represented by
$$H_i(1, 3, 5) = R_1^{i+5}R_1^{i+3}R_1^{i+1}R_7^i + \overline{H}_i(1, 3, 5) \qquad (i \geq 0).$$

Here $\overline{H}_i(1, 2, 3)$, for example, is the sum of the remaining
$$R_{j_1}^{i_1}R_{j_2}^{i_2}R_{j_3}^{i_3}R_{j_4}^{i_4} \in (\overline{1, 2, 3})$$
with $i = \min\{i_1, i_2, i_3, i_4\}$.

Theorem 3.3 [17]. (1) *The elements in* (11) *together with* h_i *and* b_j^i *form a* $\mathbb{Z}_2$*-base for the indecomposable elements in* $H^{s,*}(X)$ *for* $s \leq 4$.

(2) *There are no indecomposable elements in* $H^{5,j}(X)$ *for* $j = 2^{i+1} + 43$ *and* $j = 2^{i+1} + 8$ *where* $i > 12$.

We thus have the following.

(12) The $\mathbb{Z}_2$-module $H^{4,*}(X)$ is generated by $h_i(1, 2, 3)$, $h_i(1, 2, 4)$, $h_i(1, 2, 5)$, $h_i(1, 3, 4)$, $h_i(1, 3, 5)$ and the following decomposable elements:

$$h_ih_jh_kh_l,\ b_j^ih_kh_l,\ h_i(1)h_jh_k,\ b_j^ib_l^k,\ b_j^ih_i(1),$$
$$h_i(1)h_j(1),\ h_i(1, 2)h_j,\ h_i(1, 3)h_j.$$

(13) The $\mathbb{Z}_2$-module $H^{5,*}(X)$ is generated by indecomposable elements and the following decomposable elements:

$$h_ih_jh_kh_lh_m,\ b_j^ih_kh_lh_m,\ h_i(1)h_jj_kh_l,\ h_i(1, 2)h_jh_k,\ h_i(1, 3)h_jh_k,$$
$$h_k(1, 2, 3)h_j,\ h_i(1, 2, 4)h_j,\ h_i(1, 2, 5)h_j,\ h_i(1, 3, 4)h_j,\ h_i(1, 3, 5)h_j,$$
$$b_j^ib_l^kh_m,\ b_j^ih_k(1)h_l,\ h_k(1)h_j(1)h_k,\ h_k(1, 2)b_k^j,\ h_i(1, 2)h_j(1),$$
$$h_i(1, 3)b_k^j,\ h_i(1, 3)h_j(1).$$

For $x \in H^{s,*}(X)$ denote by $|x|$ the internal degree of x. We have the following internal degrees of the indecomposable elements in (3.3)(1).

$$(14)\quad \begin{aligned} &|h_i| = 2^i, \qquad |b_j^i| = 2^{i+1}(2^j - 1) \quad (j \geq 2), \qquad |h_i(1)| = 2^{i+3} + 2^i, \\ &|h_i(1, 2)| = 2^{i+5} + 2^{i+4} + 2^i, \qquad |h_i(1, 3)| = 2^{i+5} + 2^{i+3} + 2^i, \\ &|h_j(1, 2, 3)| = 2^{i+7} + 2^{i+6} + 2^{i+5} + 2^i, \\ &|h_i(1, 2, 4)| = 2^{i+7} + 2^{i+6} + 2^{i+4} + 2^i, \\ &|h_i(1, 2, 5)| = 2^{i+7} + 2^{i+5} + 2^{i+4} + 2^i, \\ &|h_i(1, 3, 4)| = 2^{i+7} + 2^{i+6} + 2^{i+3} + 2^i, \\ &|h_i(1, 3, 5)| = 2^{i+7} + 2^{i+5} + 2^{i+3} + 2^i. \end{aligned}$$

From this we see the decomposable elements in (12) and (13) have the following internal degrees. We note again that $j \geq 2$ when considering b_j^i.

(15)

$$\begin{aligned} &|h_ih_jh_kh_l| = 2^i + 2^j + 2^k + 2^l, &&|b_j^ih_kh_l| = 2^{i+1}(2^j - 1) + 2^k + 2^l, \\ &|h_i(1)h_jh_k| = 2^{i+3} + 2^i + 2^j + 2^k, &&|b_j^ib_l^k| = 2^{i+1}(2^j - 1) + 2^{k+1}(2^l - 1), \\ &|b_j^ih_k(1)| = 2^{i+1}(2^j - 1) + 2^{k+3} + 2^k, &&|h_i(1)h_j(1)| = 2^{i+3} + 2^i + 2^{j+3} + 2^j, \\ &|h_i(1, 2)h_j| = 2^{i+5} + 2^{i+4} + 2^i + 2^j, &&|h_i(1, 3)h_j| = 2^{i+5} + 2^{i+3} + 2^i + 2^j. \end{aligned}$$

$$(16)\quad \begin{aligned} &|h_ih_jh_kh_lh_m| = 2^i + 2^j + 2^k + 2^l + 2^m, \\ &|b_j^ih_kh_lh_m| = 2^{i+1}(2^j - 1) + 2^k + 2^l + 2^m, \\ &|h_i(1)h_jh_kh_l| = 2^{i+3} + 2^i + 2^j + 2^k + 2^l, \\ &|h_i(1, 2)h_jh_k| = 2^{i+5} + 2^{i+4} + 2^i + 2^j + 2^k, \\ &|h_i(1, 3)h_jh_k| = 2^{i+5} + 2^{i+3} + 2^i + 2^j + 2^k, \\ &|h_i(1, 2, 3)h_j| = 2^{i+7} + 2^{i+6} + 2^{i+5} + 2^i + 2^j, \\ &|h_i(1, 2, 4)h_j| = 2^{i+7} + 2^{i+6} + 2^{i+4} + 2^i + 2^j, \\ &|h_i(1, 2, 5)h_j| = 2^{i+7} + 2^{i+5} + 2^{i+4} + 2^i + 2^j, \\ &|h_i(1, 3, 4)h_j| = 2^{i+7} + 2^{i+6} + 2^{i+3} + 2^i + 2^j, \\ &|h_i(1, 3, 5)h_j| = 2^{i+7} + 2^{i+5} + 2^{i+3} + 2^i + 2^j, \\ &|b_j^ib_l^kh_m| = 2^{i+1}(2^j - 1) + 2^{k+1}(2^l - 1) + 2^m, \\ &|b_j^ih_k(1)h_l| = 2^{i+1}(2^j - 1) + 2^{k+3} + 2^k + 2^l, \\ &|h_i(1)h_j(1)h_k| = 2^{i+3} + 2^i + 2^{j+3} + 2^j + 2^k, \\ &|h_i(1, 2)b_k^j| = 2^{i+5} + 2^{i+4} + 2^i + 2^{j+1}(2^k - 1), \\ &|h_i(1, 2)h_j(1)| = 2^{i+5} + 2^{i+4} + 2^i + 2^{j+3} + 2^j, \\ &|h_i(1, 3)b_k^j| = 2^{i+5} + 2^{i+3} + 2^i + 2^{j+1}(2^k - 1), \\ &|h_i(1, 3)h_j(1)| = 2^{i+5} + 2^{i+3} + 2^i + 2^{j+3} + 2^j. \end{aligned}$$

Now assume $i > 12$. The internal degrees we are concerned with for Propositions 1.4 and 1.5 are $2^{i+1}+7$ and $2^{i+1}+42$ for $s=4$ and $2^{i+1}+8$ and $2^{i+1}+43$ for $s=5$. We have the dyadic expansions $2^{i+1}+7=2^{i+1}+2^2+2^1+2^0$, $2^{i+1}+42=2^{i+1}+2^5+2^3+2^1$, $2^{i+1}+8=2^{i+1}+2^3$ and $2^{i+1}+43=2^{i+1}+2^5+2^3+2^1+2^0$. Note that $i \gg 5$. It is a tedious but not difficult work to compare these dyadic expansions with those of the internal degrees in (14), (15) and (16) to obtain the following result, the proof of which is left to the reader. Before stating the result we recall the following relations in $H^{*,*}(X)$ [21].

$$
(17)\qquad
\begin{aligned}
&\text{(i) } h_i h_{i+1} = 0 && (i \geq 0).\\
&\text{(ii) } h_{i+2} b_2^i = h_i h_i(1) && (i \geq 0).\\
&\text{(iii) } b_2^i b_2^{i+2} = h_i^2 b_3^{i+1} + h_{i+3}^2 b_3^i && (i \geq 0).\\
&\text{(iv) } h_{i+2} h_i(1) = h_i b_2^{i+1} && (i \geq 0).\\
&\text{(v) } h_{i+3} h_i(1) = 0 && (i \geq 0).
\end{aligned}
$$

Lemma 3.4. *Assume* $i > 12$. *Then*

(1) $H^{4,2^{i+1}+42}(X)$ *is generated by* $h_{i+1}h_5h_3h_1$.

(2) $H^{5,2^{i+1}+43}(X)$ *is generated by* $h_{i+1}h_1h_0(1,3)$, $h_{i+1}h_5h_1h_0(1)$ *and* $h_{i+1}h_5h_3h_1h_0 = 0$.

(3) $H^{4,2^{i+1}+7}(X)$ *is generated by* $h_3h_0b_i^0$, $b_i^0h_0(1)$ *and* $h_{i+1}h_2h_1h_0 = 0$.

(4) $H^{5,2^{i+1}+8}(X)$ *is generated by the following elements*:

$$
\begin{aligned}
&h_2b_i^0b_2^0 = h_0b_i^0h_0(1),\ h_ib_{i-1}^1b_2^1,\ h_5b_{i-3}^2b_{i-4}^3,\ h_2b_{i-2}^2b_2^1,\ h_ib_{i-4}^3b_2^2,\\
&h_i^2h_1b_2^0,\ h_{i-1}^2h_4b_{i-3}^2,\ h_{i+1}h_0^2b_2^0,\ h_2b_{i-4}^4h_2(1) = h_4b_{i-4}^4b_2^2,\\
&h_ib_{i-6}^5h_3(1),\ h_lh_2^2b_{i-l+1}^{l-1}\ (2 \leq l \leq i-1,\ l \neq 3),\\
&h_ih_lh_3b_{i-l}^{l-1}\ (1 \leq l \leq i-2,\ l \neq 2,4),\ h_l^2h_3b_{i-l}^l\ (0 \leq l \leq i-2,\ l \neq 2,4),\\
&h_1b_{i-2}^2b_3^0,\ h_3b_3^{i-4}h_{i-3}(1),\ h_4(b_{i-2}^1)^2,\ h_{i-1}^4h_3,\ h_{i+1}h_1^4,\ h_ih_{i-1}h_4b_{i-4}^2 = 0,\\
&h_6b_{i-6}^6h_3(1) = 0,\ h_{i+1}h_2h_1h_0^2 = 0,\ h_ih_{i-1}h_{i-2}^2h_3 = 0,\\
&(h_{i-1})^2h_ih_2^2 = 0,\ h_i^2h_2h_1^2 = 0.
\end{aligned}
$$

Recall that $H^{*,*}(X)$ is the E_2-term of the May spectral sequence for $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_2, \mathbb{Z}_2)$ with convergence given by (10). The classes $h_5h_3h_1$, $h_5h_1h_0(1)$ in $H^{*,*}(X)$ are known to persist to E_∞ and represent respectively the classes $h_5h_3h_1$ and h_5c_0 in $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_2, \mathbb{Z}_2)$ [22]. It is proved in [13] that for $i > 12$, $h_{i+1}h_5h_3h_1$ and $h_{i+1}h_5c_0$ are nonzero in $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_2, \mathbb{Z}_2)$. Thus proof of Propositions 1.4 and 1.5 will be complete if we can show that $h_{i+1}h_1h_0(1,3)$ in (3.4) (2) and the nonzero elements in (3.4) (3), (4) do not persist to E_∞, and this we prove in the following context where d_2 and d_4 are differentials in the May spectral sequence.

Lemma 3.5. (1) $d_2\colon H^{5,2^{i+1}+43}(X) \to H^{6,2^{i+1}+43}(X)$ *is nonzero on* $h_{i+1}h_1h_0(1,3)$.

(2) $d_2\colon H^{4,2^{i+1}+7}(X) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to H^{5,2^{i+1}+7}(X)$ *is one-to-one.*

(3) $\ker(d_2\colon H^{5,2^{i+1}+8}(X) \to H^{6,2^{i+1}+8}(X))$ *is generated by the following elements*:

(a) $h_4(b^1_{i-2})^2$, (b) $h^4_{i-1}h_3$, (c) $h_{i+1}h_1^4$, (d) $h_ib^5_{i-6}h_3(1) + h_ih_5h_3b^4_{i-5}$,

(e) $h_ib^3_{i-4}b^2_2 + h_4b^4_{i-4}b^2_2 + h_4h^2_2b^3_{i-3} + h_ih^2_3b^2_{i-3}$,

(f) $h_ib^1_{i-2}b^1_2 + h_2b^2_{i-2}b^1_2 + h_ih_3h_1b^0_{i-1} + h^3_2b^1_{i-1}$,

(g) $h_ih_lh_3b^{l-1}_{i-l} + h^2_lh_3b^l_{i-6}$ *for* $1 \le l \le i-2$, $l \ne 2, 4$.

(4) $h_ih_4(b^1_{i-3}) \in H^{6,2^{i+1}+8}(X)$ *is nonzero, persists to* E_4 *and*

$$d_4[h_4(b^1_{i-2})^2] = h_ih_4(b^1_{i-3})^2.$$

(5) *Except* $h_4(b^1_{i-2})^2$, *all the elements in* (3) *lie in*

$$\mathrm{im}(d_2\colon H^{4,2^{i+1}+8}(X) \to H^{5,2^{i+1}+8}(X)).$$

We recall [21] there are the following differentials in the May spectral sequence.

$$d_r(h_j) = 0 \quad \text{all } r.$$
$$d_2(b^j_2) = h^3_{j+1} + h_{j+2}h^2_j.$$
$$d_2(b^j_k) = h_{j+1}b^{j+1}_{k-1} + h_{j+k}b^j_{k-1} \quad \text{for } k \ge 3.$$
$$d_2(h_j(1)) = h^2_{j+2}h_j.$$
$$d_2(h_j(1,3)) = h^2_{j+4}h_j(1) + h_{j+2}h_jh_{j+2}(1).$$
$$d_4((b^j_k)^2) = h_{j+2}(b^{j+1}_{k-1})^2 + h_{j+k+1}(b^j_{k-1})^2 \quad \text{for } k \ge 3.$$

From these we calculate relevant differentials on $h_{i+1}h_1h_0(1,3)$ and the elements in (34) (3), (4) as follows.

(18) $$d_2h_{i+1}h_1h_0(1,3) = h_{i+1}h^2_4h_1h_0(1).$$

(19.1) $$d_2h_3h_0b^0_i = h_ib_3h_0b^0_{i-1}.$$

(19.2) $$d_2b^0_ih_0(1) = h_ib^0_{i-1}h_0(1) + h_1b^1_{i-1}h_0(1).$$

(20.1) $$d_2h_2b^0_ib^0_2 = h^2_2h^2_0b^0_i + h_ih_2b^0_{i-1}b^0_2.$$

(20.2) $$d_2h_5b^2_{i-3}b^3_{i-4} = h_5h_3(b^3_{i-4})^2 + h_{i-1}h_5b^3_{i-4}b^2_{i-4} + h_{i-1}h_5b^2_{i-3}b^3_{i-5}.$$

(20.3) $$d_2h^2_ih_1b^0_2 = h^2_ih^4_1.$$

(20.4) $$d_2h^2_{i-1}h_4b^2_{i-3} = h^3_{i-1}h_4b^2_{i-4}.$$

(20.5) $$d_2 h_{i+1} h_0^2 b_2^0 = h_{i+1} h_2 h_0^4.$$

(20.6) $$d_2 h_l h_2^2 b_{i-l+1}^{l-1} = \begin{cases} h_l^2 h_2^2 b_{i-l}^l + h_i h_l h_2^2 b_{i-l}^{l-1} & \text{for } 5 \le l \le i-2, \\ h_{i-1}^4 h_2^2 & \text{for } l = i-1. \end{cases}$$

(20.7) $$\begin{aligned} d_2 h_i h_l h_3 b_{i-l}^{l-1} &= d_2 h_l^2 h_3 b_{i-l}^l \\ &= \begin{cases} h_i h_l^2 h_3 b_{i-l-1}^l & \text{for } 6 \le l \le i-3, \\ h_i h_{i-2}^4 h_3 & \text{for } l = i-2. \end{cases} \end{aligned}$$

(20.8) $$d_2 h_3 h_0^2 b_i^0 = h_i h_3 h_0^2 b_{i-1}^0.$$

(20.9) $$\begin{aligned} d_2 h_i b_{i-6}^5 h_3(1) &= h_i h_5^2 h_3 b_{i-6}^5 + h_i h_6 b_{i-7}^6 h_3(1) \\ &= h_i h_5^2 h_3 b_{i-6}^5 \quad (\text{by } (17)(\text{v})) \\ &= d_2 h_i h_5 h_3 b_{i-5}^4 = d_2 h_5^2 h_3 b_{i-5}^5. \end{aligned}$$

(This is the case $l = 5$ skipped in (20.7).)

(20.10) $$\begin{aligned} d_2 h_4 b_{i-4}^4 b_2^2 &= h_4^2 h_2^2 b_{i-4}^4 + h_i h_4 b_{i-5}^4 b_2^2 + h_5 h_2 b_{i-5}^5 h_2(1) \\ &= h_4^2 h_2^2 b_{i-4}^4 + h_i h_4 b_{i-5}^4 b_2^2 \quad (\text{by } (17)(\text{v})). \end{aligned}$$

(20.11) $$d_2 h_4 h_2^2 b_{i-3}^3 = h_4^2 h_2^2 b_{i-4}^4 + h_i h_4 h_2^2 b_{i-4}^3.$$

(This is the case $l = 4$ skipped in (20.6).)

(20.12) $$d_2 h_i b_{i-4}^3 b_2^2 = h_i h_3^3 b_{i-4}^3 + h_i h_4 h_2^2 b_{i-4}^3 + h_i h_4 b_{i-5}^4 b_2^2.$$

(20.13) $$d_2 h_i h_3^2 b_{i-3}^2 = d_2 h_3^3 b_{i-3}^3 = h_i h_3^3 b_{i-4}^3.$$

(This is the case $l = 3$ skipped in (20.7).)

(20.14) $$d_2 h_i b_{i-2}^1 b_2^1 = h_i h_2^3 b_{i-2}^1 + h_i h_2 b_{i-3}^2 b_2^1 + h_i h_3 h_1^2 b_{i-2}^1.$$

(20.15) $$d_2 h_2 b_{i-2}^2 b_2^1 = h_i h_2 b_{i-3}^2 b_2^1 + h_2^4 b_{i-2}^2.$$

(20.16) $$d_2 h_i h_3 h_1 b_{i-1}^0 = d_2 h_3 h_1^2 b_{i-1}^1 = h_i h_3 h_1^2 b_{i-2}^1.$$

(This is the case $l = 1$ skipped in (20.7).)

(20.17) $$d_2 h_2^3 b_i^1 = h_i h_2^3 b_{i-2}^1 + h_2^4 b_{i-2}^2.$$

(This is the case $l = 2$ skipped in (20.6).)

(20.18) $$d_2 h_1 b_{i-2}^2 b_3^0 = h_i h_1 b_{i-3}^2 b_3^0 + h_3 h_1 b_{i-3}^3 b_3^0 + h_3 h_1 b_{i-2}^2 b_2^0 + h_1^2 b_{i-2}^2 b_2^1.$$

(20.19)
$$\begin{aligned} d_2h_3b_3^{i-4}h_{i-3}(1) &= h_{i-1}^2h_{i-3}h_3b_3^{i-4} + h_{i-3}h_3b_2^{i-3}h_{i-3}(1) \\ &\quad + h_{i-1}h_3h_2^{i-4}h_{i-3}(1) \\ &= h_{i-1}h_3(b_2^{i-3})^2 \quad \text{(by (17)(ii), (iii), (iv))}. \end{aligned}$$

(20.20)
$$d_4h_4(b_{i-2}^1)^2 = h_ih_4(b_{i-3}^1)^2.$$

To prove Lemma 3.5 we start with (3.5) (3) and (3.5) (5). Let

$$K = \ker(d_2\colon H^{5,2^{i+1}+8}(X) \to H^{6,2^{i+1}+8}(X)).$$

It is clear that $h_{i-1}^4h_3$, $h_{i+1}h_1^4$ and $h_4(b_{i-2}^1)^2$ lie in K. These are the elements in (3.5) (3) (a), (b), (c). Let $\overline{d}$, $\overline{e}$, $\overline{f}$, $\overline{g}_l$ denote respectively the elements in (3.5) (3) (d), (e), (f) and (g). $\overline{d} \in K$ follows from (20.9). By taking the sum of (20.10) through (20.13) we see $\overline{e} \in K$. By taking the sum of (20.14) through (20.17) we see $\overline{f} \in K$. From (20.7), (20.9), (20.13) and (20.16) we see each $\overline{g}_l \in K$ for $1 \le l \le i-2$, $l \ne 2, 4$. This proves a half of (3.5) (3). (3.5) (5) follows from the following differentials.

$$d_2(h_{i-1}h_3b_2^{i-2}) = h_{i-1}^4h_3.$$
$$d_2(h_{i+1}h_1b_2^0) = h_{i+1}h_1^4.$$
$$d_2(b_{i-5}^5h_3(1) + h_5h_3b_{i-4}^4) = h_ib_{i-6}^5h_3(1) + h_ih_5h_3b_{i-5}^4.$$
$$d_2(b_{i-3}^3b_2^2 + h_3^2b_{i-2}^2) = h_4b_{i-4}^4b_2^2 + h_4h_2^2b_{i-3}^3 + h_ib_{i-4}^3b_2^2 + h_ih_3^2b_{i-3}^2.$$
$$d_2(b_{i-1}^1b_2^1 + h_3h_1b_i^0) = h_ib_{i-1}^1b_2^1 + h_2b_{i-2}^2b_2^1 + h_ih_3h_1b_{i-1}^0 + h_2^3b_i^1.$$
$$d_2(h_lh_3b_{i+1-l}^{l-1}) = h_ih_lh_3b_{i-l}^{l-1} + h_l^2h_3b_{i-l}^l, \quad \text{for } 1 \le l \le i-2,\ l \ne 2, 4.$$

To prove the other half of (3.5) (3) and the conclusions (1), (2), (4) in the Lemma we need only show the following.

(23)
(i) $h_{i+1}h_4^2h_1h_0(1) \ne 0$ in $H^{6,2^{i+1}+43}(X)$.
(ii) $h_ih_3h_0b_{i-1}^0$ and $h_ib_{i-1}^0h_0(1) + h_1b_{i-1}^1h_0(1)$ are linearly independent in $H^{5,2^{i+1}+7}(X)$.
(iii) The element $h_ih_4(b_{i-3}^1)^2$ in (20.20) and the elements in (20.1) through (20.19), except (20.10) and (20.14), are linearly independent in $H^{6,2^{i+1}+8}(X)$. (The element in (20.1), for example, means the element $d_2h_2b_i^0b_2^0$.)

To prove (23) we make an observation. Suppose given a monomial $y = (R_{j_1}^{i_1})^{r_1}\cdots(R_{j_n}^{i_n})^{r_n} \in X$ and a class $\alpha \in H^{*,*}(X)$. Any representing cycle x of α in X is a sum of monomials. We write $y \in \alpha$ to mean that there exists a representing cycle x such that y appears in the sum x. If a monomial such as y above satisfies

(*)
$$i_a \ne i_b + j_b \quad \text{for all } a, b$$

then we say this monomial satisfies the "nonboundary" condition. Suppose y satisfies $(*)$. Then the property "$y \in \alpha$" depends only on the class α, that is, if $\alpha = \{x\}$ and y appears in the cycle x then it appears in any other representing cycle x'. This follows straightforwardly from the differential formula (9). Thus we may write $y \notin \beta$ if y does not appear in some (and hence any) representing cycle of β. The observation is the following which also follows straightforwardly from (9).

(24) Let $\{\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_n\}$ be a finite subset of $H^{*,*}(X)$ with $\{\beta_1, \beta_2, \dots, \beta_n\}$ linearly independent. Suppose there exist monomials $y_1, y_2, \dots, y_m$ all satisfying $(*)$ such that $y_j \in \alpha_j$ for $1 \le j \le m$, $y_j \notin \alpha_k$ for $j \ne k$ and $y_j \notin \beta_l$ for $1 \le j \le m$, $1 \le l \le n$. Then $\{\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \dots, \beta_n\}$ is linearly independent.

We apply (24) to prove (23). First we prove (23)(i) and (23)(ii). For (23)(i) we take $m = 1$, $n = 0$ and $\alpha_1 = h_{i+1}h_4^2h_1h_0(1)$. Recall that $\{R_1^1R_3^0 + R_2^1R_2^0\} = h_0(1)$. Thus $R_1^{i+1}(R_1^4)^2(R_1^1)^2R_3^0 \in h_{i+1}h_4^2h_0(1)$. Since $i > 12$, the monomial $R_1^{i+1}R_1^4(R_1^1)^2R_3^0$ satisfies $(*)$. So $h_{i+1}h_4^2h_1h_0(1) \ne 0$. For (23)(ii) take $m = 2$, $n = 0$, $\alpha_1 = h_1h_3h_0b_{i-3}^0$ and $\alpha_2 = h_ib_{i-1}^0h_0(1) + h_1b_{i-1}^1h_0(1)$. Let $y_1 = R_1^iR_1^3R_1^0(R_{i-3}^0)$, $y_2 = R_1^i(R_{i-1}^0)R_1^1R_3^0$. Both y_1 and y_2 satisfy $(*)$. Since $y_j \in \alpha_j$, $j = 1, 2$, $y_1 \notin \alpha_2$, $y_2 \notin \alpha_1$ it follows that α_1 and α_2 are linearly independent. This proves (23)(i) and (23)(ii).

For (23)(iii) we define α_j for $1 \le j \le 2i+4$, $j \ne 10$, $i+8$, $i+10$ and $\beta_1, \beta_2, \beta_3$ as follows. For $1 \le j \le 5$, let α_j denote the element in (20.j). (We stress again that the element in (20.1), for example, means the element $d_2h_2b_i^0b_2^0$.) Let $\alpha_6, \alpha_7, \alpha_8$ denote respectively the elements in (20.8), (20.15) and (20.20). Let $\alpha_{7+l} = d_2h_lh_2^2b_{i-l+1}^{l-1}$ for $2 \le l \le i-1$, $l \ne 3$ and let $\alpha_{i+6+l} = d_2h_ih_lh_3b_{i-l}^{l-1}$ for $1 \le l \le i-2$, $l \ne 2, 4$. The elements α_{7+l} for $5 \le l \le i-1$ are in (20.6). The elements α_{i+6+l} for $6 \le l \le i-2$ are in (20.7). $\alpha_9, \alpha_{11}, \alpha_{i+7}, \alpha_{i+9}, \alpha_{i+11}$ are elements in (20.17), (20.11), (20.16), (20.13), (20.9) respectively. Let β_1 denote the element in (20.18), β_2 the element in (20.19) and β_3 the element

$$h_ih_4b_{i-5}^4b_2^2 + h_ih_4h_2^2b_{i-4}^3$$

which is the sum of the element in (20.11) and the element in (20.12). To prove (23)(iii) is equivalent to proving $\{\alpha_1, \alpha_2, \dots, \alpha_{2i+4}, \beta_1, \beta_2, \beta_3\}$ is linearly independent.

Consider the following monomials y_j for $1 \le j \le 2i+4$, $j \ne 10$, $i+8$, $i+10$.

$$y_1 = (R_1^2)^2(R_1^0)^2(R_i^0)^2, \qquad y_2 = R_1^5R_1^3(R_{i-4}^3)^4,$$
$$y_3 = (R_1^i)^2(R_1^1)^4, \qquad y_4 = (R_1^{i-1})^3R_1^4(R_{i-4}^2)^2,$$

$$y_5 = R_1^{i+1} R_1^2 (R_1^0)^4, \qquad y_6 = R_1^i R_1^3 (R_1^0)^2 (R_{i-1}^0)^2,$$
$$y_7 = R_1^i R_1^2 (R_{i-3}^2)(R_2^1)^2, \qquad y_8 = R_1^i R_1^4 (R_{i-3}^1)^4,$$
$$y_{7+l} = \begin{cases} (R_1^l)^2 (R_1^2)^2 (R_{i-l}^l)^2 & \text{for } 4 \le l \le i-2, \\ R_1^i (R_2^2)^3 (R_{i-2}^1)^2 & \text{for } l = 2, \\ (R_1^{i-1})^4 (R_1^2)^2 & \text{for } l = i-1, \end{cases}$$
$$y_{i+6+l} = \begin{cases} R_1^i (R_1^l)^2 R_1^3 (R_{i-l-1}^l)^2 & \text{for } 1 \le l \le i-3, \ l \ne 2, 4, \\ R_1^i (R_1^{i-2})^4 R_1^3 & \text{for } l = i-2. \end{cases}$$

Since $i > 12$, each y_j satisfies the "nonboundary" condition $(*)$. One checks that $\{y_1, y_2, \ldots, y_{2i+4}\}$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_{2i+4}, \beta_1, \beta_2, \beta_3\}$ have the relations described in (24). Thus $\{\alpha_1, \alpha_2, \ldots, \alpha_{2i+4}, \beta_1, \beta_2, \beta_3\}$ is linearly independent if we can show the following.

(25) $\{\beta_1, \beta_2, \beta_3)$ is linearly independent.

We recall that

$$\beta_1 = h_i h_1 b_{i-3}^2 b_3^0 + h_3 h_1 b_{i-3}^3 b_3^0 + h_3 h_1 b_{i-2}^2 b_2^0 + h_1^2 b_{i-2}^2 b_2^1,$$
$$\beta_2 = h_{i-1} h_3 (b_2^{i-3})^2, \quad \text{and}$$
$$\beta_3 = h_i h_4 h_2^2 b_{i-4}^3 + h_i h_4 b_{i-5}^4 b_2^2.$$

They are represented respectively by the following cycles in $X^{6, 2^{i+1}+6}$.

$$\begin{aligned} &R_1^i R_1^1 (R_{i-3}^2)^2 (R_3^0)^2 + R_1^3 R_1^1 (R_{i-3}^3)^2 (R_3^0)^2 + R_1^3 R_1^1 (R_{i-2}^2)^2 (R_2^0)^2 \\ &\qquad + (R_1^1)^2 (R_{i-2}^2)^2 (R_2^1)^2, \\ &R_1^{i-1} R_1^3 (R_2^{i-3})^4, \\ &R_1^i R_1^4 (R_1^2)^2 (R_{i-4}^3)^2 + R_1^i R_1^4 (R_{i-5}^4)^2 (R_2^2)^2. \end{aligned} \tag{26}$$

We prove (25) by working in the $\mathbb{Z}_2$-dual $\mathrm{Hom}(X, \mathbb{Z}_2) = Y$ which is a differential coalgebra over $\mathbb{Z}_2$ with differential given by dualizing the differential (9) in X. Let $(P_{j_1}^{i_1})^{r_1} \cdots (P_{j_n}^{i_n})^{r_n}$ be the element in Y dual to $(R_{j_1}^{i_1})^{r_1} \cdots (R_{j_n}^{i_n})^{r_n}$ in the monomial basis for X. One verifies that the elements

$$\begin{aligned} x_1 &= P_1^i P_1^1 (P_{i-3}^2)^2 (P_3^0)^2 + P_1^i P_1^{i-2} P_{i-3}^1 P_{i-3}^2 (P_3^0)^2, \\ x_2 &= P_1^{i-1} P_1^3 (P_2^{i-3})^4 + P_1^{i-2} P_1^{i-3} P_1^3 P_3^{i-3} (P_2^{i-3})^2, \\ x_3 &= P_1^i P_1^4 (P_1^2)^2 (P_{i-4}^3)^2 + P_1^i P_1^4 (P_{i-7}^6)^2 (P_4^2)^2 \\ &\qquad + P_1^i P_1^4 P_1^2 P_{i-4}^3 P_{i-7}^6 P_4^2 \end{aligned} \tag{27}$$

are cycles in $Y^{6, 2^{i+1}+8}$. Let $\gamma_j = \{x_j\}$, $j = 1, 2, 3$. Comparing (26) and (27) we see $\langle \gamma_j, \beta_k \rangle = \delta_{jk}$. Thus we need only show $\{\gamma_1, \gamma_2, \gamma_3\}$ is linearly independent in $H^{6, 2^{i+1}+8}(Y)$.

Let $\Delta: Y \to Y \otimes Y$ be the coproduct. The component element in $Y^{4,2^i+16} \otimes Y^{2,2^i-8}$ of $\Delta(x_1)$ (resp. $\Delta(x_2), \Delta(x_3)$) is $P_1^i P_1^1 (P_3^0)^2 \otimes (P_{i-3}^2)^2$ (resp. $0, 0$). The component element in $Y^{2,2^{i-1}+8} \otimes Y^{4,2^i+2^{i-1}}$ of $\Delta(x_2)$ (resp. $\Delta(x_1), \Delta(x_3)$) is $P_1^{i-1} P_1^3 \otimes (P_2^{i-3})^4$ (resp. $0, 0$). The component element in $Y^{3,2^i+8} \otimes Y^{3,2^i}$ of $\Delta(x_3)$ (resp. $\Delta(x_1), \Delta(x_2)$) is $P_1^i (P_1^2)^2 \otimes P_1^4 (P_{i-4}^3)^2$ (resp. $0, 0$). Each of $P_i^{i-1} P_1^3$, $(P_2^{i-3})^4$, $P_1^i (P_1^2)^2$, $P_1^4 (P_{i-4}^3)^2$, $P_1^i P_1^1 (P_3^0)^2$, $(P_{i-3}^2)^2$ is a cycle. Their duals $R_1^{i-1} R_1^3$, $(R_2^{i-3})^4$, $R_1^i (R_1^2)^2$, $R_1^4 (R_{i-4}^3)^2$, $R_1^i R_1^1 (R_3^0)^2$, $(R_{i-3}^2)^2$ are also cycles in X representing nontrivial cohomology classes by (24) since they all satisfy $(*)$. Thus $\{\Delta_*(\gamma_1), \Delta_*(\gamma_2), \Delta_*(\gamma_3)\}$ is linearly independent; so $\{\gamma_1, \gamma_2, \gamma_3\}$ is linearly independent. This proves (23).

This completes the proof of Lemma 3.5 and therefore Proposition 1.4 and Proposition 1.5.

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